Indian Statistical Institute, Bangalore
M. Math First Year
Second Semester - Complex Analysis

Mid-term Exam — Date: February 24, 2020
Maximum marks: 30 — Duration: 2 hours

Answer any three and each question carries 10 Marks

1. (i)Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence R. Prove that f is holomorphic and $f'(z) = \sum_{n=0}^{\infty} na_n(z-a)^{n-1}$ has radius of convergence R.

   (ii) Determine all analytic functions f such that $\overline{f}$ is also analytic (*Marks: 3*).

   (iii) Determine all analytic functions f such that $\mathrm{Re(z)}f(z)$ is also analytic (*Marks: 2*).

2. (i) Determine the region in which the following function is defined and holomorphic
$$f(z) = \int_0^1 \frac{1}{1+tz}dt. \qquad (Marks : 5)$$

   (ii) For any closed path γ and $z \notin \mathbb{C} \setminus \gamma^*$, prove that $\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}$ is an integer.

3. Let $\phi \in H(\Omega)$ and $\phi'(z_0) \neq 0$ for some $z_0 \in \Omega$.

   (i) Prove that ϕ has a holomorphic inverse in a neighbourhood of z_0.

   (ii) If $g = f \odot \phi$ and f has a zero of order m at $\phi(z_0)$, prove that g has a zero at z_0 of order m (*Marks: 4*).

4. (i) Let $P_r(t)$ be the Poisson kernel. Prove that $P_r(\theta - t) = \mathrm{Re}(\frac{e^{it}+z}{e^{it}-z})$ where $z = re^{i\theta}$, $\int_{-\pi}^{\pi} P_r(t)dt = 2\pi$, $P_r(t) > 0$ and $P_r(\delta) \to 0$ uniformly in $\delta \in [t, \pi]$ as $r \to 1$ for any $t > 0$ (*Marks: 7*).

   (ii) Determine all harmonic functions u such that u^2 is harmonic.

5. (i) Prove that a harmonic function that is zero on T is zero on U also (DO NOT USE any result) (*Marks: 5*).

   (ii) Let (u_n) be a sequence of positive harmonic functions on a region Ω. Suppose $u_n(z_0) \to 0$ for some $z_0 \in \Omega$. Describe the behaviour of (u_n) in the rest of Ω.